Generalized Perfectly Matched Layer—An Extension of Berenger's Perfectly Matched Layer Boundary Condition

Jiayuan Fang, *Member, IEEE,* and Zhonghua Wu, *Student Member, IEEE*

***Abstract*—Berenger's perfectly matched layer (PML) has been found very effective in absorbing propagating waves, but it is ineffective in absorbing evanescent waves. Also, since the impedance of PML does not match those of most of lossy media, the PML technique can generally not be applied to terminate lossy media. Derived from the modified Maxwell's equations in the stretched coordinates, the absorber, which we call the generalized perfectly matched layer (GPML), presented in this letter cannot only absorb propagating waves, but also accelerate the attenuation of evanescent waves and perfectly match arbitrary lossy media. Verifications of GPML are provided with numerical examples.**

I. Introduction

BERENGER'S perfectly matched layer (PML) boundary condition has been successfully applied in the finite-difference time-domain (FDTD) computation for open-region electromagnetic scattering and radiation problems [1], [2]. Extensive numerical tests have demonstrated that the PML can absorb outgoing propagating waves very effectively, resulting in reduced requirements on computer memory space and CPU times for many problems. It has also been shown that the PML cannot effectively absorb evanescent waves [3], [4]. To see this point clearly, consider a plane wave in a PML medium with conductivities $\sigma_y = \sigma_z = 0$. Assume the angles between the wave propagation direction and the x and the z axes are φ and $90°$, respectively. Let ψ be any field component of the plane wave, then ψ can be expressed as ((15) of [1])

$$\psi = \psi_0 e^{j\omega(t-(x\cos\varphi+y\sin\varphi/c))} e^{-(\sigma_x \cos\varphi/\epsilon c)x}. \quad (1)$$

For a propagating wave in the x direction, cos φ is a real number and the term $e^{-\sigma_x \cos\varphi/(\epsilon c)x}$ contributes to the attenuation of the field. On the other hand, if the variation of the field in the x direction is of evanescent nature, $\cos\varphi$ will be an imaginary number. In that case, the term $e^{-\sigma_x \cos\varphi/(\epsilon c)x}$ no longer decays in the x direction and the presence of the PML does not add any additional attenuation to the wave.

Another limitation of the PML technique is that it can only be applied to PML-PML interfaces [1]. A necessary condition to be a PML medium is [1]

$$\sigma/\epsilon = \sigma^*/\mu. \quad (2)$$

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The authors are with the Department of Electrical Engineering, the State University of New York at Binghamton, Binghamton, NY 13902-6000 USA.

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If the medium on either one side of the interface is not a PML, significant reflection may occur and the corresponding reflection coefficient can be found in (28) of [1]. This limitation excludes the application of PML to many problems involving lossy media, since (2) is not satisfied for most lossy media.

A modification of PML is proposed in [3] for the absorption of evanescent waves, where media in the interior computation domains are lossless. No reports have appeared, to our knowledge, on remedies of the PML so that it can be applied for general lossy media. This letter presents the formulation of a modified PML medium, which we call the generalized perfectly matched layer (GPML), by using the modified Maxwell's equations with the stretched coordinates. It will be shown that the GPML can be used to terminate interior regions of both lossless and lossy (nonPML) media and effectively absorb both propagating and evanescent waves simultaneously.

II. Formulation of the Generalized Perfectly Matched Layer

In a medium of parameters $(\epsilon, \mu, \sigma, \sigma^*)$, where σ and σ^* may not be related by (2), the modified Maxwell's equations in the stretched coordinates can be expressed as

$$\nabla_e \times \vec{E} = -j\omega\mu'\vec{H}, \quad \nabla_n \times \vec{H} = j\omega\epsilon'\vec{E} \quad (3)$$

where $\mu' = \mu + \sigma^*/j\omega$ and $\epsilon' = \epsilon + \sigma/j\omega$, and

$$\nabla_e = \vec{a}_x \frac{1}{e_x}\frac{\partial}{\partial x} + \vec{a}_y \frac{1}{e_y}\frac{\partial}{\partial y} + \vec{a}_z \frac{1}{e_z}\frac{\partial}{\partial z} \quad (4)$$

$$\nabla_h = \vec{a}_x \frac{1}{h_x}\frac{\partial}{\partial x} + \vec{a}_y \frac{1}{h_y}\frac{\partial}{\partial y} + \vec{a}_z \frac{1}{h_z}\frac{\partial}{\partial z} \quad (5)$$

where $e_i, h_i, i = x, y, z$ are coordinate stretching variables. Following the same procedure as that in [5] and imposing the boundary condition for a plane wave solution at a two-medium interface at $(x = 0)$, one can find that the reflection coefficients for both TE to x- and TM to x-polarized plane waves are zero for any arbitrary values of σ and σ^* if $\epsilon_1' = \epsilon_2', \mu_1' = \mu_2', e_{1y} = e_{2y} = h_{1y} = h_{2y}, e_{1z} = e_{2z} = h_{1z} = h_{2z}, e_{1x} = h_{1x}$, and $e_{2x} = h_{2x}$. Assume the medium on one side of the interface is in the interior region and $(e_{1x}, e_{1y}, e_{1z}, h_{1x}, h_{1y}, h_{1z}) = (1, 1, 1, 1, 1, 1)$. Then, the medium on the other side of the interface which acts as an absorber should have $(e_{2x}, e_{2y}, e_{2z}, h_{2x}, h_{2y}, h_{2z}) =$

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$(s_x, 1, 1, s_x, 1, 1)$. s_x is the ratio of e_{2x} and e_{1x}, and can be chosen as the following form:

$$s_x(x) = s_0(x)\left[1 - j\frac{\sigma_x(x)}{\omega\epsilon}\right]. \quad (6)$$

With s_x chosen as (6), the plane wave solution in the GPML absorber can be derived to be

$$\psi = \psi_0 e^{j(\omega t - k\sin\phi y)} e^{-j(k'_x - (k''_x\sigma_x/\omega\epsilon))s_0 x} e^{-(k''_x + (k'_x\sigma_x/\omega\epsilon))s_0 x} \quad (7)$$

where $k^2 = \omega^2\mu'\epsilon'$, and $k \cos\phi = k'_x - jk''_x$. The absorber thus constructed can absorb both propagating and evanescent waves. If the plane wave is mainly a propagating wave, i.e., k'_x is dominant, the amplitude of the wave decays as $e^{-(k'_x\sigma_x/\omega\epsilon)s_0 x}$, which is similar to that in the PML. If the plane wave is evanescent in the x direction, i.e., k''_x is dominant, the wave decays mainly as $e^{-k''_x s_0 x}$, whereas in the original PML, the evanescent wave decays as $e^{-k''_x x}$. By choosing $s_0 > 1$, the attenuation of the evanescent wave is accelerated.

III. Implementation of the Generalized Perfectly Matched Layer

For the convenience of illustration, consider the TM to x wave. The same procedure can be applied to TE to x and arbitrary 3-D cases. The governing field equations in GPML can be derived to the following forms:

$$\frac{\partial E_y}{s_0\partial x} = -\mu\frac{\partial H_{zx}}{\partial t} - (\sigma^* + \sigma_x^*)H_{zx} - \psi_{zx} \quad (8)$$

$$\frac{\partial E_x}{\partial y} = \mu\frac{\partial H_{zy}}{\partial t} + \sigma^* H_{zy} \quad (9)$$

$$\frac{\partial H_z}{s_0\partial x} = -\epsilon\frac{\partial E_y}{\partial t} - (\sigma + \sigma_x)E_y - \phi_y \quad (10)$$

$$\frac{\partial H_z}{\partial y} = \epsilon\frac{\partial E_x}{\partial t} + \sigma E_x \quad (11)$$

$$\frac{\partial \psi_{zx}}{\partial t} = \frac{\sigma^*\sigma_x^*}{\mu}H_{zx} \quad (12)$$

$$\frac{\partial \phi_y}{\partial t} = \frac{\sigma\sigma_x}{\epsilon}E_y \quad (13)$$

where $\sigma_x^* = \sigma_x\mu/\epsilon, \psi_{zx}$, and ϕ_y are two auxiliary variables to facilitate the implementation. When $s_0 = 1$, and σ and σ^* equal to zero, (8)–(11) become the equations for the original PML [1]. The discretization of (8)–(13) can be realized by the standard central difference in space and time.

IV. Selection of $s_0(x)$ and $\sigma_x(x)$

Although the GPML theoretically perfectly matches any lossless and lossy interior media, $s_0(x)$ and $\sigma_x(x)$ have to change gradually and continuously to avoid significant numerical reflections as a wave travels through the absorber. We found following patterns of $s_0(x)$ and $\sigma_x(x)$ can generally result in small numerical reflections

$$s_0(x) = 1 + s_m\left(\frac{x}{\delta}\right)^2 \quad (14)$$

$$\sigma_x(x) = \sin^2\left(\frac{\pi x}{2\delta}\right) \quad (15)$$

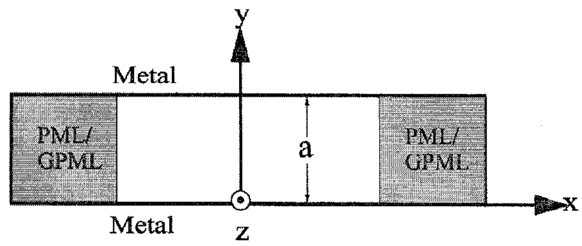

Fig. 1. Cross-section of a parallel-plate waveguide (a = 40 mm).

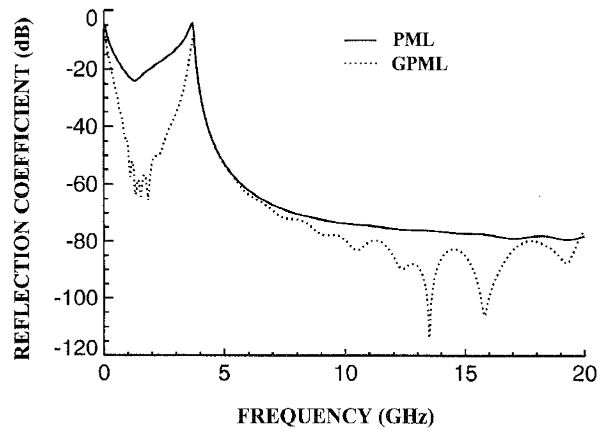

Fig. 2. Reflection coefficient of 16-cell PML and GPML for $R_0 = 10^{-4}$.

where s_m is a coefficient and δ is the absorber thickness. As been discussed before, a propagating wave attenuates in GPML in a rate proportional to $e^{-\sigma_x(x)s_0(x)}$. The selection of $\sigma_x(x)$ and $s_0(x)$ in (14) and (15) makes the variation of $\sigma_x(x)s_0(x)$ a parabolic function almost uniformly in the entire region of the GPML absorber.

The wavelength of an incident wave shrinks as the wave penetrates into the absorber. The numerical reflection becomes significant if the spatial resolution of the wave is too small. We found that s_m needs to be bounded by the condition $\lambda/s_m > 2 - 3\ dh$, where dh is the FDTD space step and λ is the wavelength in the interior medium terminated by the GPML.

V. Numerical Examples

Consider a parallel-plate waveguide filled with free space, as shown in Fig. 1. The separation between the two metal plates is 40 mm. The FDTD space step is chosen to be 1 mm. The TM_1 mode of the waveguide has a cutoff frequency at 3.75 GHz. The reflection coefficients of a 16-cell PML and a 16-cell GPML for a TM_1 incident wave are shown in Fig. 2. As can be seen from Fig. 2, the PML is only effective in the frequency range above the cutoff frequency, while the GPML cannot only absorb the propagating wave, but also add a substantial damping to the evanescent wave. In this test, the theoretical reflection coefficients R_0 for a normally incident plane wave is set to 10^{-4} for both PML and GPML media.

Consider the waveguide shown in Fig. 1 again, but this time the waveguide is filled with a lossy medium of $\sigma = 0.1$ S/m, and $\sigma^* = 0$. Since the lossy medium is not a PML, an absorber with conductive losses σ_x and σ_x^* only, i.e., without

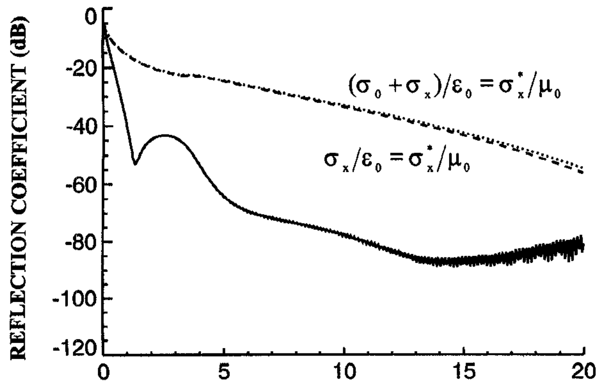

Fig. 3. Reflection coefficient of TM_1 mode for a parallel plate waveguide filled with a lossy material ($\sigma = 0.1$ S/m, $\sigma^* = 0$). $R_0 = 10^{-4}$. (Solid line is obtained with GPML.)

the auxiliary variables defined in (12) and (13), cannot absorb effectively. Fig. 3 shows the reflection coefficients of two absorbers of different selections of σ_x and σ_x^*, without the auxiliary variables ψ_{zx} and ϕ_y. On the other hand, very good absorption can be achieved by the GPML implemented with (8)–(13).

Applications of the GPML have also been carried out for modeling wave propagation along microstrip and strip lines, as well as in layered lossy media. Similar observations have been obtained from these applications as from those described above.

VI. Conclusion

The generalized perfectly matched layer, which is an extension of Berenger's perfectly matched layer, is presented in this letter. The new absorber (GPML) can perfectly match nonPML lossy media and effectively absorb both propagating and evanescent waves.

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